Quasi-Monte Carlo sampling for stochastic variational problems

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Introduction

- Computational methods for solving stochastic variational problems require (first) a discretization of the underlying probability distribution induced by a numerical integration scheme for the approximate computation of expectations and (second) an efficient solver for a (large scale) finite-dimensional variational problem.

- Discretization means scenario or sample generation.

- **Standard approach:** Variants of Monte Carlo (MC) methods.

- **Recent alternative approaches to scenario generation:**
  
  (a) **Optimal quantization of probability distributions**
      
      (Pflug-Pichler 11).

  (b) **Quasi-Monte Carlo (QMC) methods**
      
      (Koivu-Pennanen 05, Pennanen 09, Homem-de-Mello 08, Heitsch-Leövey-Römisch 12).

  (c) **Sparse grid quadrature rules**
      
      (Chen-Mehrotra 08).
Known convergence rates in terms of scenario or sample size $n$:

- **MC**: $\hat{e}_n(f) = O(n^{-\frac{1}{2}})$ if $f \in L_2$,
- (a): $e_n(f) = O(n^{-\frac{1}{d}})$ if $f \in \text{Lip}$,
- (b): classical: $e_n(f) = O(n^{-1}(\log n)^{d})$ if $f \in \text{BV}$,
  
  recently: $\hat{e}_n(f) \leq C(\delta)n^{-1+\delta} \,(\delta \in (0, \frac{1}{2}])$ if $f \in W^{(1,\ldots,1)}$,
  
  where $C(\delta)$ does not depend on $d$,
- (c): $e_n(f) = O(n^{-r}(\log n)^{(d-1)(r+1)})$ if $f \in W^{(r,\ldots,r)}$,

where $d$ is the dimension of the random vector and $e_n(f)$ the quadrature error for integrand $f$ and sample size $n$, i.e.,

$$e_n(f) = \left| \int_{[0,1]^d} f(\xi) d\xi - \frac{1}{n} \sum_{i=1}^{n} f(\xi^i) \right|$$

and $\hat{e}_n(f)$ denotes mean (square) quadrature error.

Monte Carlo methods and (a) may be justified by available stability results for stochastic programs, but there is almost no reasonable justification for (b) and (c) in many cases.

In applications of stochastic programming dimension $d$ is often large.
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**Quasi-Monte Carlo methods**

We consider the approximate computation of

\[ I_d(f) = \int_{[0,1]^d} f(\xi) d\xi \]

by a QMC algorithm

\[ Q_{n,d}(f) = \frac{1}{n} \sum_{i=1}^{n} f(\xi^i) \]

with (non-random) points \( \xi^i, i = 1, \ldots, n \), from \([0,1]^d\).

We assume that \( f \) belongs to a linear normed space \( \mathbb{F}_d \) of functions on \([0,1]^d\) with norm \( \| \cdot \|_d \) and unit ball \( \mathbb{B}_d \).

Worst-case error of \( Q_{n,d} \) over \( \mathbb{B}_d \):

\[ e(Q_{n,d}) = \sup_{f \in \mathbb{B}_d} |I_d(f) - Q_{n,d}(f)| \]
Koksma-Hlawka type inequalities: (Koksma-Hlawka 61)

\[ e_n(f) = |I_d(f) - Q_n,d(f)| \leq \|\text{disc}\|_{p,p'}\|f\|_{q,q'}, \]

where \(1 \leq p, p', q, q' \leq \infty, \frac{1}{p} + \frac{1}{q} = 1, \frac{1}{p'} + \frac{1}{q'} = 1,\) and

\[ \|\text{disc}\|_{p,p'} = \left( \sum_{u \subseteq D} \gamma_u \left( \int_{[0,1]^{\lvert u \rvert}} \left| \text{disc}(x_u, 1) \right|^{p'} dx_u \right)^{\frac{p}{p'}} \right)^{\frac{1}{p}} \]

\[ \text{disc}(x) = \prod_{j=1}^{d} x_j - \frac{1}{n} \sum_{i=1}^{n} 1_{[0,x)}(\xi^i) \quad (x \in [0,1)^d) \]

\[ \|f\|_{q,q'} = \left( \sum_{u \subseteq D} \gamma_u^{-1} \left( \int_{[0,1]^{\lvert u \rvert}} \left| \frac{\partial |u|f}{\partial x_u}(x_u, 1) \right|^{q'} dx_u \right)^{\frac{q}{q'}} \right)^{\frac{1}{q}} \]

with the obvious modifications if one or more of \(p, p', q, q'\) are infinite.

By \((x_u, 1)\) we mean the \(d\)-dimensional vector with the same components as \(x\) for indices in \(u\) and the rest of the components replaced by 1.

In particular, the classical Koksma-Hlawka inequality essentially corresponds to \(p = p' = \infty\) if \(f\) belongs to the tensor product Sobolev space \(\mathcal{W}_{2,\gamma,\text{mix}}^{(1,...,1)}([0, 1]^d)\) which is defined next.
The case of kernel reproducing Hilbert spaces

We assume that $\mathbb{F}_d$ is a **kernel reproducing Hilbert space** with inner product $\langle \cdot, \cdot \rangle$ and kernel $K : [0, 1]^d \times [0, 1]^d \to \mathbb{R}$, i.e.,

$$K(\cdot, y) \in \mathbb{F}_d \text{ and } \langle f(\cdot), K(\cdot, y) \rangle = f(y) \quad (\forall y \in [0, 1]^d, f \in \mathbb{F}_d).$$

If $I_d$ is a linear bounded functional on $\mathbb{F}_d$, the quadrature error $e_n(Q_{n,d})$ allows the representation

$$e_n(Q_{n,d}) = \sup_{f \in \mathbb{B}_d} |I_d(f) - Q_{n,d}(f)| = \sup_{f \in \mathbb{B}_d} |\langle f, h_n \rangle| = \|h_n\|_d$$

according to Riesz’ theorem for linear bounded functionals.

The **representer** $h_n \in \mathbb{F}_d$ of the quadrature error is of the form

$$h_n(x) = \int_{[0,1]^d} K(x, y)dy - \frac{1}{n} \sum_{i=1}^{n} K(x, \xi^i) \quad (\forall x \in [0, 1]^d),$$

and it holds

$$e_n^2(Q_{n,d}) = \int_{[0,1]^{2d}} K(x, y)dx \, dy - \frac{2}{n} \sum_{i=1}^{n} \int_{[0,1]^d} K(\xi^i, y)dy + \frac{1}{n^2} \sum_{i,j=1}^{n} K(\xi^i, \xi^j)$$

(Hickernell 98)
Example: Weighted tensor product Sobolev space

\[ \mathbb{F}_d = \mathcal{W}_{2,\gamma,\text{mix}}^{(1,\ldots,1)}([0, 1]^d) = \bigotimes_{i=1}^d W_2^1([0, 1]) \]

equipped with the weighted norm \( \|f\|_{\gamma}^2 = \langle f, f \rangle_{\gamma} \) and inner product

\[ \langle f, g \rangle_{\gamma} = \sum_{u \subseteq \{1,\ldots,d\}} \gamma_u^{-1} \int_{[0,1]^{|u|}} \frac{\partial^{|u|} f}{\partial x_u}(x_u, 1) \frac{\partial^{|u|} g}{\partial x_u}(x_u, 1) dx_u , \]

where \( \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_d > 0, \gamma_u = \prod_{j \in u} \gamma_j \), is a kernel reproducing Hilbert space with the kernel

\[ K_{d,\gamma}(x, y) = \prod_{j=1}^d (1 + \gamma_j \mu(x_j, y_j)) \quad (x, y \in [0, 1]^d), \]

where

\[ \mu(t, s) = \begin{cases} \min\{|t - 1|, |s - 1|\} , & (t - 1) (s - 1) > 0, \\ 0 , & \text{else.} \end{cases} \]

Note that \( f \in \mathbb{F}_d \) iff \( \frac{\partial^{|u|} f}{\partial x_u} (\cdot, 1) \in L_2([0, 1]^{|u|}) \) for all \( u \subseteq D \).
Theorem: (Sloan-Woźniakowski 98)

Let $F_d = \mathcal{W}_{2,\gamma,\text{mix}}^{(1,\ldots,1)}([0,1]^d)$. Then the worst-case error

$$e^2(Q_{n,d}) = \sup_{\|f\|_\gamma \leq 1} |I_d(f) - Q_{n,d}(f)| = \sum_{\emptyset \neq u \subseteq D} \prod_{j \in u} \gamma_j \int_{[0,1]^u} \text{disc}^2(x_u, 1) dx_u$$

is called weighted $L_2$-discrepancy of $\xi^1, \ldots, \xi^n$.

Problem: Integrands of stochastic variational problems are typically piecewise smooth and do not belong to $F_d$ in general (piecewise linear convex functions are even not of bounded variation (Owen 05)).

Typical integrands: $f = g \circ h = g(h(\cdot))$, where $g$ is piecewise linear-quadratic (convex) and $h$ is sufficiently smooth.

First results for $g(t) = \max\{0, t\}$ and $h$ smooth via the ANOVA decomposition (Griebel-Kuo-Sloan 10, 13)

Here: Integrands in linear two-stage stochastic programming, i.e., maximum of linear-quadratic functions.
First general QMC construction: **Digital nets** (Sobol 69, Niederreiter 87)

Let $m, t \in \mathbb{Z}_+, m > t$.

A set of $b^m$ points in $[0, 1)^d$ is a $(t, m, d)$-net in base $b$ if every elementary subinterval $E = \prod_{j=1}^{d} \left[ \frac{a^j}{b^d_j}, \frac{a^j+1}{b^d_j} \right)$ in base $b$ with $\lambda^d(E) = b^{t-m}$ contains $b^t$ points.

A sequence $(\xi^i)$ in $[0, 1)^d$ is a $(t, d)$-sequence in base $b$ if, for all integers $k \in \mathbb{Z}_+$ and $m > t$, the set

$$\{ \xi^i : kb^m \leq i < (k + 1)b^m \}$$

is a $(t, m, d)$-net in base $b$.

There exist $(t, d)$-sequences $(\xi^i)$ in $[0, 1]^d$ such that $e_n(f) = O(n^{-1}(\log n)^{d-1})$.

**Specific sequences:**
Faure, Sobol’, Niederreiter, Niederreiter-Xing sequences (Dick-Pillichshammer 10).

Second general QMC construction: **Lattices** (Korobov 59, Sloan-Joe 94)

Let $g \in \mathbb{Z}^d$ and consider the lattice points

$$\{ \xi^i = \{ i \over n g \} : i = 1, \ldots, n \},$$

where $\{ z \} = z - [z] \in [0, 1)$ is the componentwise fractional part. The generator $g$ is chosen such that the lattice rule has good convergence properties.
Fig. 5.3 Four different point sets with $n = 64$: random (top left), rectangular grid (top right), Korobov lattice (bottom left), and Sobol' (bottom right).
Recent development: Randomly scrambled \((t, m, d)\)-nets (Owen 95) and randomized lattice rules (Sloan-Kuo-Joe 02).

Randomly shifted lattice points:
With independent in \([0, 1]^d\) uniformly distributed \(\triangle_i, i = 1, \ldots, n\), put

\[
Q_{n,d}(f) = \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n} g + \triangle_i\right).
\]

Theorem:
Let \(n\) be prime, \(\mathbb{F}_d = \mathcal{W}_{2,\gamma,\text{mix}}([0, 1]^d)\) and \(g \in \mathbb{Z}^d\) be constructed componentwise. Then there exists for any \(\delta \in (0, \frac{1}{2}]\) a constant \(C(\delta) > 0\) such that the mean quadrature error attains the optimal convergence rate

\[
\hat{e}(Q_{n,d}) \leq C(\delta) n^{-1+\delta},
\]

where the constant \(C(\delta)\) grows when \(\delta\) decreases, but does not depend on the dimension \(d\) if the sequence \((\gamma_j)\) satisfies the condition

\[
\sum_{j=1}^{\infty} \gamma_j^{\frac{1}{2(1-\delta)}} < \infty \quad \text{(e.g. } \gamma_j = \frac{1}{j^3}).
\]
ANOVA decomposition of multivariate functions

Idea: Use decompositions of $f$, where the terms are smooth or small.
Let $D = \{1, \ldots, d\}$ and $f \in L_{1, \rho}(\mathbb{R}^d)$ with $\rho(\xi) = \prod_{j=1}^d \rho_j(\xi_j)$, where for $p \geq 1$

$f \in L_{p, \rho}(\mathbb{R}^d)$ iff $\int_{\mathbb{R}^d} |f(\xi)|^p \rho(\xi) d\xi < \infty$ iff $\int_{(0,1)^d} |g(t)|^p dt < \infty$

$g = f \circ \Phi^{-1}$, $\Phi^{-1} := (\Phi_1^{-1}, \ldots, \Phi_d^{-1})$ and $\Phi_j(x_j) := \int_{-\infty}^{x_j} \rho_j(\xi_j) d\xi_j, \; j \in D$.

Let the projection $P_k$ and $P_k^*$, $k \in D$, be defined by

$$(P_k f)(\xi) := \int_{-\infty}^\infty f(\xi_1, \ldots, \xi_{k-1}, s, \xi_{k+1}, \ldots, \xi_d) \rho_k(s) ds \quad (\xi \in \mathbb{R}^d).$$

$$(P_k^* g)(t) := \int_0^1 g(t_1, \ldots, t_{k-1}, s, t_{k+1}, \ldots, t_d) ds \quad (t \in (0,1)^d).$$

For $u \subseteq D$ we write

$$P_uf = \left( \prod_{k \in u} P_k \right)(f) \quad \text{and} \quad P_u^* g = \left( \prod_{k \in u} P_k^* \right)(g),$$

where the product means composition, and note that the ordering within the product is not important because of Fubini’s theorem.
The functions $P_u f$ and $P^*_u g$ are constant with respect to all $\xi_k$ and $t_k$, $k \in u$.

**ANOVA-decomposition of $f$:**

$$f = \sum_{u \subseteq D} f_u, \quad g = \sum_{u \subseteq D} g_u \quad \text{and} \quad g_u(t_u) = f_u \circ \Phi^{-1}_u(t_u) \quad (t_u \in (0, 1)^{|u|}),$$

where $f_{\emptyset} = I_d(f) = P_D(f)$ and recursively

$$f_u = P_{-u}(f) - \sum_{v \subset u} f_v \quad \text{and} \quad g_u = P^*_{-u}(g) - \sum_{v \subset u} g_v$$

or according to (Kuo-Sloan-Wasilkowski-Woźniakowski 10)

$$f_u = \sum_{v \subseteq u} (-1)^{|u|-|v|} P_{-v}f = P_{-u}(f) + \sum_{v \subseteq u} (-1)^{|u|-|v|} P_{u-v}(P_{-u}(f)),$$

where $P_{-u}$ and $P_{u-v}$ mean integration with respect to $\xi_j$, $j \in D \setminus u$ and $j \in u \setminus v$, respectively. This motivates that $f_u$ is essentially as smooth as $P_{-u}(f)$.

If $f$ belongs to $L_{2,\rho}(\mathbb{R}^d)$, its **ANOVA terms $\{f_u\}_{u \subseteq D}$ are orthogonal in $L_{2,\rho}(\mathbb{R}^d)$**.

We set $\sigma^2(f) = \| f - I_d(f) \|^2_{L_2}$ and $\sigma^2_u(f) = \| f_u \|^2_{L_2}$, and have

$$\sigma^2(f) = \| f \|^2_{L_2} - (I_d(f))^2 = \sum_{\emptyset \neq u \subseteq D} \sigma^2_u(f).$$
Owen’s superposition (truncation) dimension distribution of $f$: Probability measure $\nu_S$ ($\nu_T$) defined on the power set of $D$

$$\nu_S(s) := \sum_{|u|=s} \frac{\sigma^2_u(f)}{\sigma^2(f)} \quad \left( \nu_T(s) = \sum_{\max \{ j : j \in u \} = s} \frac{\sigma^2_u(f)}{\sigma^2(f)} \right) \quad (s \in D).$$

Effective superposition (truncation) dimension $d_S(\varepsilon)$ ($d_T(\varepsilon)$) of $f$ is the $(1 - \varepsilon)$-quantile of $\nu_S$ ($\nu_T$):

$$d_S(\varepsilon) = \min \left\{ s \in D : \sum_{|u| \leq s} \sigma^2_u(f) \geq (1 - \varepsilon) \sigma^2(f) \right\} \leq d_T(\varepsilon)$$

$$d_T(\varepsilon) = \min \left\{ s \in D : \sum_{u \subseteq \{1,\ldots,s\}} \sigma^2_u(f) \geq (1 - \varepsilon) \sigma^2(f) \right\}$$

It holds

$$\max \left\{ \| f - \sum_{|u| \leq d_S(\varepsilon)} f_u \|_{2,\rho}, \| f - \sum_{u \subseteq \{1,\ldots,d_T(\varepsilon)\}} f_u \|_{2,\rho} \right\} \leq \sqrt{\varepsilon} \sigma(f).$$

(Caflisch-Morokoff-Owen 97, Owen 03, Wang-Fang 03)
Two-stage linear stochastic programs

We consider the linear two-stage stochastic program

\[
\min \left\{ \int_{\Xi} f(x, \xi) P(d\xi) : x \in X \right\},
\]

where \( f \) is extended real-valued defined on \( \mathbb{R}^m \times \mathbb{R}^d \) given by

\[
f(x, \xi) = \langle c, x \rangle + \Phi(q(\xi), h(\xi) - T(\xi)x), (x, \xi) \in X \times \Xi,
\]

\( c \in \mathbb{R}^m, X \subseteq \mathbb{R}^m \) and \( \Xi \subseteq \mathbb{R}^d \) are convex polyhedral, \( W \) is an \((r, \overline{m})\)-matrix, \( P \) is a Borel probability measure on \( \Xi \), and the vectors \( q(\xi) \in \mathbb{R}^{\overline{m}}, h(\xi) \in \mathbb{R}^r \) and the \((r, m)\)-matrix \( T(\xi) \) are affine functions of \( \xi \), \( \Phi \) is the second-stage optimal value function on \( \mathbb{R}^{\overline{m}} \times \mathbb{R}^r \)

\[
\Phi(u, t) = \inf \{ \langle u, y \rangle : W y = t, y \geq 0 \} = \max \{ \langle t, z \rangle : W^\top z \leq u \},
\]

Let \( \text{pos} \ W = W(\mathbb{R}^{\overline{m}}_+), \ D = \{ u \in \mathbb{R}^{\overline{m}} : \{ z \in \mathbb{R}^r : W^\top z \leq u \} \neq \emptyset \} \).

Assumptions:

\( \textbf{(A1)} \) \( h(\xi) - T(\xi)x \in \text{pos} W \) and \( q(\xi) \in D \) for all \( (x, \xi) \in X \times \Xi \).

\( \textbf{(A2)} \) \( \int_{\Xi} \| \xi \|^2 P(d\xi) < \infty. \)
**Proposition:** (A1) and (A2) imply that the two-stage stochastic program represents a convex minimization problem with respect to the first stage decision $x$ with polyhedral constraints.

**Lemma:** (Walkup-Wets 69, Nožička-Guddat-Hollatz-Bank 74) $\Phi$ is finite, polyhedral and continuous on the $(\overline{m} + r)$-dimensional polyhedral cone $D \times \text{pos } W$ and there exist $(r, \overline{m})$-matrices $C_j$ and $(\overline{m} + r)$-dimensional polyhedral cones $K_j$, $j = 1, \ldots, \ell$, such that

$$\bigcup_{j=1}^{\ell} K_j = D \times \text{pos } W \quad \text{and} \quad \text{int } K_i \cap \text{int } K_j = \emptyset, \ i \neq j,$$

$$\Phi(u, t) = \langle C_j u, t \rangle, \quad \text{for each } (u, t) \in K_j, \ j = 1, \ldots, \ell.$$

The function $\Phi(u, \cdot)$ is convex on $\text{pos } W$ for each $u \in D$, and $\Phi(\cdot, t)$ is concave on $D$ for each $t \in \text{pos } W$. The intersection $K_i \cap K_j, i \neq j$, is either equal to $\{0\}$ or contained in a $(\overline{m} + r - 1)$-dimensional subspace of $\mathbb{R}^{\overline{m}+r}$ if the two cones are adjacent.
**ANOVA decomposition of two-stage integrands**

**Assumptions:**
(A1), (A2) and
(A3) $P$ has a density of the form $\rho(\xi) = \prod_{j=1}^{d} \rho_j(\xi_j)$ ($\xi \in \mathbb{R}^d$) with continuous marginal densities $\rho_j$, $j \in D$.

**Proposition:**
(A1) implies that the function $f(x, \cdot)$, where

$$f_x(\xi) := f(x, \xi) = \langle c, x \rangle + \Phi(q(\xi), h(\xi) - T(\xi)x) \quad (x \in X, \xi \in \Xi)$$

is the two-stage integrand, is continuous and piecewise linear-quadratic. For each $x \in X$, $f(x, \cdot)$ is linear-quadratic on each polyhedral set

$$\Xi_j(x) = \{ \xi \in \Xi : (q(\xi), h(\xi) - T(\xi)x) \in K_j \} \quad (j = 1, \ldots, \ell).$$

It holds $\text{int} \Xi_j(x) \neq \emptyset$, $\text{int} \Xi_j(x) \cap \text{int} \Xi_i(x) = \emptyset$, $i \neq j$, and the sets $\Xi_j(x)$, $j = 1, \ldots, \ell$, decompose $\Xi$. Furthermore, the intersection of two adjacent sets $\Xi_i(x)$ and $\Xi_j(x)$, $i \neq j$, is contained in some $(d-1)$-dimensional affine subspace.
To compute projections $P_k f$ for $k \in D$, let $\xi_i \in \mathbb{R}$, $i = 1, \ldots, d$, $i \neq k$, be given. We set $\xi^k = (\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_d)$ and

$$\xi_k(s) = (\xi_1, \ldots, \xi_{k-1}, s, \xi_{k+1}, \ldots, \xi_d) \in \mathbb{R}^d \quad (s \in \mathbb{R}).$$

We fix $x \in X$ and consider the one-dimensional affine subspace $\{\xi_k(s) : s \in \mathbb{R}\}$:

Example with $d = 2 = p$, where the polyhedral sets are cones

It meets the nontrivial intersections of two adjacent polyhedral sets $\Xi_i(x)$ and $\Xi_j(x)$, $i \neq j$, at finitely many points $s_i$, $i = 1, \ldots, p$ if all $(d - 1)$-dimensional subspaces containing the intersections do not parallel the $k$th coordinate axis.
The \( s_i = s_i(\xi^k), \ i = 1, \ldots, p, \) are affine functions of \( \xi^k. \) It holds

\[
s_i = - \sum_{l=1, l \neq k}^{p} \frac{g_{il}}{g_{ik}} \xi_l + a_i \quad (i = 1, \ldots, p)
\]

for some \( a_i \in \mathbb{R} \) and \( g_i \in \mathbb{R}^d \) belonging to an intersection of polyhedral sets.

**Proposition:**

Let \( k \in D, \ x \in X. \) Assume (A1)–(A3) and that all \((d - 1)\)-dimensional affine subspaces containing nontrivial intersections of adjacent sets \( \Xi_i(x) \) and \( \Xi_j(x) \) do not parallel the \( k \)th coordinate axis.

Then the \( k \)th projection \( P_k f \) has the explicit representation

\[
P_k f(\xi^k) = \sum_{i=1}^{p+1} \sum_{j=0}^{2} p_{ij}(\xi^k; x) \int_{s_{i-1}}^{s_i} s^j \rho_k(s) ds,
\]

where \( s_0 = -\infty, \ s_{p+1} = +\infty \) and \( p_{ij}(\cdot; x) \) are polynomials in \( \xi^k \) of degree \( 2-j, \) \( j = 0, 1, 2, \) with coefficients depending on \( x, \) and is continuously differentiable. \( P_k f \) is infinitely differentiable if the marginal density \( \rho_k \) belongs to \( C^\infty(\mathbb{R}). \)
**Theorem:**

Let $x \in X$, assume (A1)–(A3) and that the following geometric condition (GC) be satisfied: All $(d - 1)$-dimensional affine subspaces containing nontrivial intersections of adjacent sets $\Xi_i(x)$ and $\Xi_j(x)$ do not parallel any coordinate axis. Then the ANOVA approximation

$$f_{d-1} := \sum_{|u| \leq d-1} f_u \quad \text{i.e.} \quad f = f_{d-1} + f_D$$

of $f$ is infinitely differentiable if all densities $\rho_k$, $k \in D$, belong to $C_b^\infty(\mathbb{R})$. Here, the subscript $b$ means that all derivatives of functions belonging to that space are bounded on $\mathbb{R}$. 
Example: Let $\bar{m} = 3, \ d = 2, \ P$ denote the two-dimensional standard normal distribution, $h(\xi) = \xi$, $q$ and $W$ be given such that (A1) is satisfied and the dual feasible set is

$$
\{z \in \mathbb{R}^2 : -z_1 + z_2 \leq 1, \ z_1 + z_2 \leq 1, -z_2 \leq 0\}.
$$

Dual feasible set, its vertices $v^j$ and the normal cones $\mathcal{K}_j$ to its vertices

The function $\Phi$ and the integrand are of the form

$$
\Phi(t) = \max_{i=1,2,3} \langle v^i, t \rangle = \max\{t_1, -t_1, t_2\} = \max\{|t_1|, t_2\}
$$

$$
f(\xi) = \langle c, x \rangle + \Phi(\xi - Tx) = \langle c, x \rangle + \max\{|\xi_1 - [Tx]_1|, \xi_2 - [Tx]_2\}
$$

and the convex polyhedral sets are $\Xi_j(x) = Tx + \mathcal{K}_j, \ j = 1, 2, 3$.
The ANOVA projection $P_1f$ is in $C^\infty$, but $P_2f$ is not differentiable.
QMC quadrature error estimates

If the assumptions of the theorem are satisfied, the two-stage integrand \( f = f_x \) (for fixed \( x \in X \)) allows the representation \( f = f_{d-1} + f_D \) with \( f_{d-1} \) belonging to \( \mathbb{F}_d \). This implies

\[
\left| \int_{[0,1]^d} f(\xi) d\xi - \frac{1}{n} \sum_{j=1}^{n} f(\xi^j) \right| \leq e(Q_{n,d}) \| f_{d-1} \|_{\gamma} + \left| \int_{[0,1]^d} f_D(\xi) d\xi - \frac{1}{n} \sum_{j=1}^{n} f_D(\xi^j) \right|
\]

\[
\leq e(Q_{n,d}) \| f_{d-1} \|_{\gamma} + \| f_D \|_{L_2} + \left( \frac{1}{n} \sum_{j=1}^{n} |f_D(\xi^j)|^2 \right)^{\frac{1}{2}}
\]

where \( \| \cdot \|_{\gamma} \) is the weighted tensor product Sobolev space norm.

As \( f_D \) is (Lipschitz) continuous and if the \( \xi^j, j = 1, \ldots, n \) are properly selected, the last term in the above estimate may be assumed to be bounded by \( 2\| f_D \|_{L_2} \).

Hence, if the effective superposition dimension satisfies \( d_S(\varepsilon) \leq d - 1 \), i.e., \( \| f_D \|_{L_2} \leq \sqrt{\varepsilon} \sigma(f) \) holds for some small \( \varepsilon > 0 \), the first term \( e(Q_{n,d}) \| f_{d-1} \|_{\gamma} \) dominates and the convergence rate of \( e(Q_{n,d}) \) becomes most important.
Question: How important is the geometric condition (GC)?

Partial answer: If $P$ is normal with nonsingular covariance matrix, (GC) is satisfied for almost all covariance matrices. Namely, it holds

Proposition: Let $x \in X$, (A1), (A2) be satisfied, $\text{dom } \Phi = \mathbb{R}^r$ and $P$ be a normal distribution with nonsingular covariance matrix $\Sigma$. Then the infinite differentiability of the ANOVA approximation $f_{d-1}$ of $f$ is a generic property, i.e., it holds in a residual set (countable intersection of open dense subsets) in the metric space of orthogonal $(d, d)$-matrices $Q$ (endowed with the norm topology) appearing in the spectral decomposition $\Sigma = Q^\top D Q$ of $\Sigma$ (with a diagonal matrix $D$ containing the eigenvalues of $\Sigma$).

Question: For which two-stage stochastic programs is $\|f_D\|_{L^2,\rho}$ small, i.e., the effective superposition dimension $d_S(\varepsilon)$ of $f$ is less than $d - 1$ or even much less?

Partial answer: In case of a (log)normal probability distribution $P$ the effective dimension depends on the mode of decomposition of the covariance matrix into a diagonal one.
Dimension reduction in case of (log)normal distributions

Let $P$ be the normal distribution with mean $\mu$ and nonsingular covariance matrix $\Sigma$. Let $A$ be a matrix satisfying $\Sigma = AA^\top$. Then $\eta$ defined by $\xi = A\eta + \mu$ is standard normal.

A universal principle is principal component analysis (PCA). Here, one uses $A = (\sqrt{\lambda_1}u_1, \ldots, \sqrt{\lambda_d}u_d)$, where $\lambda_1 \geq \cdots \geq \lambda_d > 0$ are the eigenvalues of $\Sigma$ in decreasing order and the corresponding orthonormal eigenvectors $u_i$, $i = 1, \ldots, d$. Wang-Fang 03, Wang-Sloan 05 report an enormous reduction of the effective truncation dimension in financial models if PCA is used.

A problem-dependent principle may be based on the following equivalence principle (Papageorgiou 02, Wang-Sloan 11).

**Proposition:** Let $A$ be a fixed $d \times d$ matrix such that $AA^\top = \Sigma$. Then it holds $\Sigma = BB^\top$ if and only if $B$ is of the form $B = AQ$ with some orthogonal $d \times d$ matrix $Q$.

**Idea:** Determine $Q$ for given $A$ such that the effective truncation dimension is minimized (Wang-Sloan 11).
Some computational experience

We considered a two-stage production planning problem for maximizing the expected revenue while satisfying a fixed demand in a time horizon with $d = T = 100$ time periods and stochastic prices for the second-stage decisions. It is assumed that the probability distribution of the prices $\xi$ is log-normal. The model is of the form

$$\max \left\{ \sum_{t=1}^{T} \left( c_t^T x_t + \int_{\mathbb{R}^T} q_t(\xi)^T y_t P(d\xi) \right) : Wy + Vx = h, y \geq 0, x \in X \right\}$$

The use of PCA for decomposing the covariance matrix has led to effective truncation dimension $d_T(0.01) = 2$. As QMC methods we used a randomly scrambled Sobol sequence (SSobol) (Owen, Hickernell) with $n = 2^7, 2^9, 2^{11}$ and a randomly shifted lattice rule (Sloan-Kuo-Joe) with $n = 127, 509, 2039$, weights $\gamma_j = \frac{1}{j^2}$ and for MC the Mersenne-Twister. 10 runs were performed for the error estimates and 30 runs for plotting relative errors.

Average rate of convergence for QMC: $O(n^{-0.9})$ and $O(n^{-0.8})$. Instead of $n = 2^7$ SSobol samples one would need $n = 10^4$ MC samples to achieve a similar accuracy as SSobol.
$\log_{10}$ of the relative errors of MC, SLA (randomly shifted lattice rule) and SSOB (scrambled Sobol’ points)
Conclusions

- Our analysis provides a theoretical basis for applying QMC methods accompanied by dimension reduction techniques to two-stage stochastic programs.
- The analysis also applies to sparse grid quadrature techniques.
The results are extendable and will be extended to mixed-integer two-stage models, multi-stage models, and to other stochastic variational problems.

Second-stage optimal value function of an integer program (van der Vlerk)
References


H. Heitsch, H. Lövey and W. Römisch, Are Quasi-Monte Carlo algorithms efficient for two-stage stochastic programs?, *Stochastic Programming E-Print Series* 5-2012 (www.speps.org) and submitted.


