Scenario generation

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Approximation issues

We consider a stochastic program of the form

$$\min \left\{ \int_{\Xi} \Phi(\xi, x) P(d\xi) : x \in X \right\},$$

where $X \subseteq \mathbb{R}^m$ is a constraint set, P a probability distribution on $\Xi \subseteq \mathbb{R}^d$, and $f = \Phi(\cdot, x)$ is a decision-dependent integrand.

Any approach to solving such models computationally requires to replace the integral by a quadrature rule

$$Q_{n,d}(f) = \sum_{i=1}^{n} w_i f(\xi^i),$$

with weights $w_i \in \mathbb{R}$ and scenarios $\xi^i \in \Xi$, $i = 1, \ldots, n$.

If the natural condition $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$ is satisfied, $Q_{n,d}(f)$ allows the interpretation as integral with respect to the discrete probability measure Q_n having scenarios ξ^i with probabilities w_i, $i = 1, \ldots, n$.

Assumption: P has a density ρ w.r.t. λ^d.

Now, we set $\mathcal{F} = \{\Phi(\cdot, x)\rho(\cdot) : x \in X\}$ and assume that the set $\mathcal{F}$ is a bounded subset of some linear normed space F_d with norm $\|\cdot\|_d$ and unit ball $\mathbb{B}_d = \{f \in F_d : \|f\|_d \leq 1\}$.

The absolute error of the quadrature rule $Q_{n,d}$ is

$$e(Q_{n,d}) = \sup_{f\in\mathbb{B}_d} \left| \int_\Xi f(\xi)d\xi - \sum_{i=1}^{n} w_i f(\xi^i) \right|$$

and the approximation criterion is based on the relative error and a given tolerance $\varepsilon > 0$, namely, it consists in finding the smallest number $n_{\min}(\varepsilon, Q_{n,d}) \in \mathbb{N}$ such that

$$e(Q_{n,d}) \leq \varepsilon e(Q_{0,d}),$$

holds, where $Q_{0,d}(f) = 0$ and, hence, $e(Q_{0,d}) = \|I_d\|$ with

$$I_d(f) = \int_\Xi f(\xi)d\xi.$$

Alternatively, we look for a suitable set $\mathcal{F}$ of functions such that $\{C\Phi(\cdot, x) : x \in X\} \subseteq \mathcal{F}$ for some constant $C > 0$ and, hence,

$$e(Q_{n,d}) \leq \frac{1}{C} \sup_{f \in \mathcal{F}} \left| \int_\Xi f(\xi) P(d\xi) - \int_\Xi f(\xi) Q_n(d\xi) \right| = D(P, Q_n),$$

and that D is a metric distance between probability distributions.

Example: Fortet-Mourier metric (of order $r \geq 1$)

$$\zeta_r(P, Q) := \sup \left| \int_\Xi f(\xi)(P - Q)(d\xi) : f \in \mathcal{F}_r(\Xi) \right|,$$

where

$$\mathcal{F}_r(\Xi) := \{f : \Xi \mapsto \mathbb{R} : f(\xi) - f(\tilde{\xi}) \leq c_r(\xi, \tilde{\xi}), \forall \xi, \tilde{\xi} \in \Xi\},$$

$$c_r(\xi, \tilde{\xi}) := \max\{1, \|\xi\|^{r-1}, \|\tilde{\xi}\|^{r-1}\} \|\xi - \tilde{\xi}\| \quad (\xi, \tilde{\xi} \in \Xi).$$

The behavior of $e(Q_{n,d})$ with respect to $n \in \mathbb{N}$ and of $n_{\min}(\varepsilon, Q_{n,d})$ with respect to ε is of considerable interest. In both cases the dependence on the dimension d of P is often crucial, too.

The behavior of both quantities depends heavily on the normed space F_d and the set $\mathcal{F}$, respectively.
It is desirable that an estimate of the form

$$n_{\min}(\varepsilon, Q_{n,d}) \leq C\, d^q \varepsilon^{-p}$$

is valid for some nonnegative constants C, q, $p > 0$ and for every $\varepsilon \in (0,1)$. Of course, $q = 0$ is highly desirable for high-dimensional problems.

Proposition: (Stability)
Let the set X be compact. Then there exists $L > 0$ such that

$$\left| \inf_{x \in X} \int_{\Xi} \Phi(\xi, x)\rho(\xi) d\xi - \inf_{x \in X} \sum_{i=1}^{n} w_i \Phi(\xi^i, x)\rho(\xi^i) \right| \leq L\, e(Q_{n,d}).$$

The solution set mapping is upper semicontinuous at P.

Examples of normed spaces F_d relevant in SP:

(a) The Banach space $F_d = \mathrm{Lip}(\mathbb{R}^d)$ of Lipschitz continuous functions equipped with the norm

$$\|f\|_d = |f(0)| + \sup_{\xi \neq \tilde{\xi}} \frac{|f(\xi) - f(\tilde{\xi})|}{\|\xi - \tilde{\xi}\|}.$$

The best possible convergence rate is $e(Q_{n,d}) = O(n^{-\frac{1}{d}})$.
It is attained for $w_i = \frac{1}{n}$ and certain ξ^i, $i = 1, \ldots, n$, if P has finite moments of order $1 + \delta$ for some $\delta > 0$. (Graf-Luschgy 00)

(b) **Assumption:** $\Xi = [0,1]^d$ (attainable by suitable transformations).
We consider the Banach space $F_d = \mathrm{BV}_{\mathrm{HK}}([0,1]^d)$ of functions having bounded variation in the sense of Hardy and Krause equipped with the norm $\|f\|_d = |f(0)| + V_{\mathrm{HK}}(f)$.
Then for $w_i = \frac{1}{n}$, $i = 1, \ldots, n$, there exist $\xi^n \in [0,1]^d$, $n \in \mathbb{N}$ such that the convergence rate is

$$e(Q_{n,d}) = O\left(\frac{(\log n)^{d-1}}{n}\right).$$

(c) The tensor product Sobolev space

$$F_{d,\gamma} = \mathcal{W}_{2,\mathrm{mix}}^{(1,\dots,1)}([0,1]^d) = \bigoplus_{i=1}^{d} W_2^1([0,1])$$

of real functions on $[0,1]^d$ having first order mixed weak derivatives with the (weighted) norm

$$\|f\|_{d,\gamma} = \left(\sum_{u \subset D} \gamma_u^{-1} \int_{[0,1]^{|u|}} \left| \frac{\partial^{|u|}}{\partial \xi^u} f(\xi^u, 1^{-u}) \right|^2 d\xi^u \right)^{\frac{1}{2}},$$

where $D = \{1, \dots, d\}$, $\gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_d > 0$, $\gamma_\emptyset = 1$ and

$$\gamma_u = \prod_{j \in u} \gamma_j \quad (u \subseteq D).$$

Note that any $f \in \mathcal{W}_{2,\mathrm{mix}}^{(1,\dots,1)}([0,1]^d)$ is of bounded variation in the sense of Hardy and Krause.

For n prime, $w_i = \frac{1}{n}$, there exist $\xi^i \in [0,1]^d$, $i = 1, \dots, n$ such that

$$e(Q_{n,d}) \le C_d(\delta) n^{-1+\delta} \|I_d\|$$

for all $0 < \delta \le \frac{1}{2}$.

Generation of a number of scenarios

We will discuss the following four scenario generation methods for stochastic programs *without nonanticipativity constraints*:

(a) Monte Carlo sampling from the underlying probability distribution P on $\mathbb{R}^d$ (Shapiro 03).

(b) Optimal quantization of probability distributions (Pflug-Pichler 10).

(c) Quasi-Monte Carlo methods (Koivu-Pennanen 05).

(d) Quadrature rules based on sparse grids (Chen-Mehrotra 08).

Monte Carlo sampling methods

Monte Carlo methods are based on drawing independent identically distributed (iid) Ξ-valued random samples $\xi^1(\cdot), \ldots, \xi^n(\cdot), \ldots$ (defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$) from an underlying probability distribution P (on Ξ) such that

$$Q_{n,d}(\omega)(f) = \frac{1}{n} \sum_{i=1}^{n} f(\xi^i(\omega)),$$

i.e., $Q_{n,d}(\cdot)$ is a random functional, and it holds

$$\lim_{n \to \infty} Q_{n,d}(\omega)(f) = \int_{\Xi} f(\xi) P(d\xi) = \mathbb{E}(f) \quad \mathbb{P}\text{-almost surely}$$

for every real continuous and bounded function f on Ξ.

If P has finite second order moments, the error estimate

$$\mathbb{E}\left(\left| \frac{1}{n} \sum_{i=1}^{n} f(\xi^i(\omega)) - \mathbb{E}(f) \right|^2 \right) \leq \frac{\mathbb{E}\left((f - \mathbb{E}(f))^2\right)}{n}$$

is valid. Hence, the mean square convergence rate is

$$\|Q_{n,d}(\omega)(f) - \mathbb{E}(f)\|_{L_2} = \sigma(f)n^{-\frac{1}{2}},$$

where $\sigma^2(f) = \mathbb{E}\left((f - \mathbb{E}(f))^2\right)$.
The latter holds without any assumption on f except $\sigma(f) < \infty$.

Remarkable property: The rate does not depend on d.

Deficiencies: (Niederreiter 92)
(i) There exist only *probabilistic error bounds*.
(ii) Possible regularity of the integrand *does not improve* the rate.
(iii) Generating (independent) random samples is *difficult*.

Practically, iid samples are approximately obtained by pseudo random number generators as uniform samples in $[0, 1]^d$ and later transformed to more general sets Ξ and distributions P.

Classical generators for pseudo random numbers are based on linear congruential methods. As the parameters of this method, we choose a large $M \in \mathbb{N}$ (*modulus*), a *multiplier* $a \in \mathbb{N}$ with $1 \leq a < M$ and $gcd(a, M) = 1$, and $c \in Z_M = \{0, 1, \ldots, M-1\}$. Starting with $y_0 \in Z_M$ a sequence is generated by

$$y_n \equiv a y_{n-1} + c \mod M \qquad (n \in \mathbb{N})$$

and the linear congruential pseudo random numbers are

$$\xi^n = \frac{y_n}{M} \in [0, 1).$$

Example: $M = 2^{32}$, $a \equiv 5 \mod 8$, and c odd (period M).

Use only pseudo random number generators having passed a series of statistical tests, e.g., uniformity test, serial correlation test, serial test, coarse lattice structure test etc.

Optimal quantization of probability measures

Let D be a metric distance of probability measures on $\mathbb{R}^d$, e.g., the Fortet-Mourier metric ζ_r of order r, or some other metric such that the underlying stochastic program behaves stable with respect to D.

Let P be a given probability distribution on $\mathbb{R}^d$. We are looking for a discrete probability measure Q_n with support $\mathrm{supp}(Q_n) = \{\xi^1, \dots, \xi^n\}$ and $Q_n(\{\xi^i\}) = \frac{1}{n}$, $i = 1, \dots, n$, such that it is the best approximation to P with respect to D, i.e.,

$$D(P, Q_n) = \min\{D(P, Q) : |\mathrm{supp}(Q)| = n, Q \text{ is uniform}\}.$$

Existence of best approximations and their convergence rates are well known for Wasserstein metrics (Graf-Luschgy 00).
Best approximations for standard normal distributions are known for $d = 1$ and $d = 2$.

In general, however, the function

$$\Psi(\xi^1, \ldots, \xi^n) := D\left(P, \frac{1}{n}\sum_{i=1}^{n} \delta_{\xi^i}\right)$$

is nonconvex and nondifferentiable on $\mathbb{R}^{dn}$. Hence, the global minimization of Ψ is not an easy task.

Algorithmic procedures for minimizing Ψ globally may be based on stochastic gradient algorithms, stochastic approximation methods and stochastic branch-and-bound techniques (e.g. Hochreiter-Pflug 07, Pflug-Pichler 10, Pagés et al 04)

Quasi-Monte Carlo methods

The basic idea of Quasi-Monte Carlo (QMC) methods is to replace random samples in Monte Carlo methods by deterministic points that are uniformly distributed in $[0,1]^d$. The latter property may be defined in terms of the so-called star-discrepancy of $\xi^1,\ldots,\xi^n$

$$D_n^*(\xi^1,\ldots,\xi^n)) := \sup_{\xi\in[0,1]^d} \left|\lambda^d([0,\xi)) - \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{[0,\xi)}(\xi^i)\right|,$$

namely, by calling a sequence $(\xi^i)_{i\in\mathbb{N}}$ uniformly distributed in $[0,1]^d$ if for $n\to\infty$

$$D_n^*(\xi^1,\ldots,\xi^n) \to 0\,.$$

A **classical result** due to Roth 54 states

$$D_n^*(\xi^1,\ldots,\xi^n) \geq B_d \frac{(\log n)^{\frac{d-1}{2}}}{n}$$

for some constant B_d and all sequences (ξ^i) in $[0,1]^d$.

There are two classical convergence results for QMC methods.

Theorem: (Proinov 88)
If the real function f is continuous on $[0,1]^d$, then there exists $C > 0$ such that

$$|Q_{n,d}(f) - I_d(f)| \le C\omega_f\Big(D_n^*(\xi^1,\ldots,\xi^n)^{\frac{1}{d}}\Big),$$

where $\omega_f(\delta) = \sup\{|f(\xi) - f(\tilde{\xi})| : \|\xi - \tilde{\xi})\| \le \delta,\ \xi, \tilde{\xi} \in [0,1]^d\}$ is the modulus of continuity of f.

Theorem: (Koksma-Hlawka 61)
If f is of bounded variation in the sense of Hardy and Krause, it holds

$$|I_d(f) - Q_{n,d}(f)| \le V_{\mathrm{HK}}(f) D_n^*(\xi_1,\ldots,\xi^n)\,.$$

for any $n \in \mathbb{N}$ and any $\xi^1,\ldots,\xi^n \in [0,1]^d$.

There exist sequences (ξ^i) in $[0,1]^d$ such that

$$D_n^*(\xi^1,\ldots,\xi^n) = O(n^{-1}(\log n)^{d-1}).$$

First general construction: (Sobol 69, Niederreiter 87)

Elementary subintervals E in base b:

$$E = \prod_{j=1}^{d} \left[\frac{a_j}{b^{d_j}}, \frac{a_j + 1}{b^{d_j}} \right),$$

with $a_i, d_i \in \mathbb{Z}_+, 0 \leq a_i < d_i, i = 1, \ldots, d$.

Let $m, t \in \mathbb{Z}_+$, $m > t$.

A set of b^m points in $[0,1]^d$ is a (t, m, d)-net in base b if every elementary subinterval E in base b with $\lambda^d(E) = b^{t-m}$ contains b^t points.

A sequence (ξ^i) in $[0,1]^d$ is a (t, d)-sequence in base b if, for all integers $k \in \mathbb{Z}_+$ and $m > t$, the set

$$\{\xi^i : kb^m \leq i < (k+1)b^m\}$$

is a (t, m, d)-net in base b.

Proposition: $(0, d)$-sequences exist if $d \leq b$.

Theorem:
The star-discrepancy of a $(0, m, d)$-net $\{\xi_i\}$ in base b satisfies

$$D_n^*(\xi_i) \leq A_d(b)\frac{(\log n)^{d-1}}{n} + O\left(\frac{(\log n)^{d-2}}{n}\right).$$

Special cases: Sobol, Faure and Niederreiter sequences.

Second general construction: (Korobov 59, Sloan-Joe 94)

Let $g \in \mathbb{Z}^d$ and consider the lattice points

$$\left\{\xi^i = \left\{\frac{i}{n}g\right\} : i = 1, \ldots, n\right\},$$

where $\{z\}$ is defined componentwise and for $z \in \mathbb{R}_+$ it is the *fractional part* of z, i.e., $\{z\} = z - \lfloor z \rfloor \in [0, 1)$.
Similar convergence results may be obtained for the star-discrepancy.

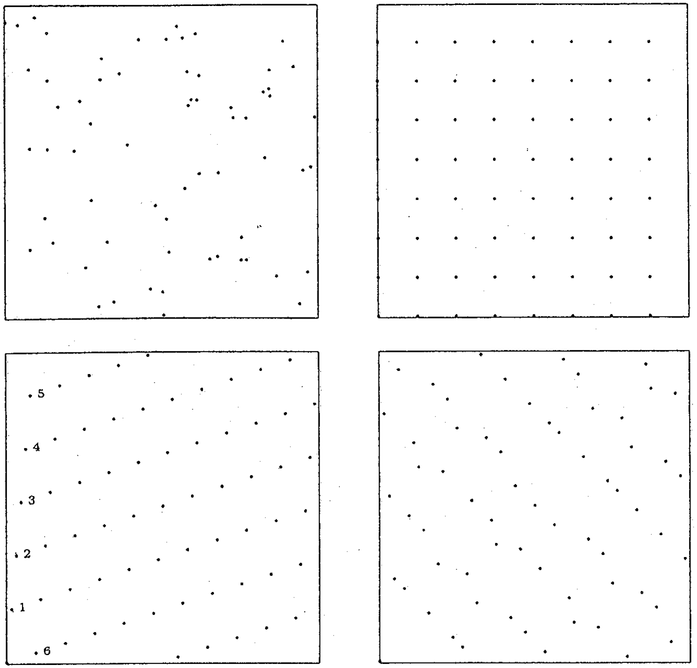


Fig. 5.3 Four different point sets with $n = 64$: random (top left), rectangular grid (top right), Korobov lattice (bottom left), and Sobol' (bottom right).

Quadrature rules with sparse grids

Again we consider the unit cube $[0,1]^d$ in $\mathbb{R}^d$. Let nested sets of grids in $[0,1]$ be given, i.e.,

$$\Xi^i = \{\xi_1^i, \ldots, \xi_{m_i}^i\} \subset \Xi^{i+1} \subset [0,1] \quad (i \in \mathbb{N}),$$

for example, the dyadic grid

$$\Xi^i = \left\{ \frac{j}{2^i} : j = 0, 1, \ldots, 2^i \right\}.$$

Then the point set suggested by Smolyak

$$H(q,d) := \bigcup_{\sum_{j=1}^d i_j = q} \Xi^{i_1} \times \cdots \times \Xi^{i_d} \qquad (q \in \mathbb{N})$$

is called a sparse grid in $[0,1]^d$. In case of dyadic grids in $[0,1]$ the set $H(q,d)$ consists of all d-dimensional dyadic grids with product of mesh size given by $\frac{1}{2^q}$.

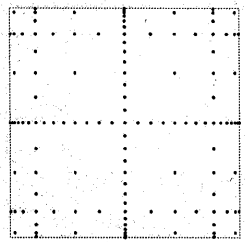
The corresponding tensor product quadrature rule for $q \geq d$ on $[0,1]^d$ with respect to the Lebesgue measure λ^d is of the form

$$Q_{q,d}(f) = \sum_{q-d+1\leq|\mathbf{i}|\leq q} (-1)^{q-|\mathbf{i}|}\binom{d-1}{q-|\mathbf{i}|}\sum_{j_1=1}^{m_{i_1}}\cdots\sum_{j_d=1}^{m_{i_d}} f(\xi_{j_1}^{i_1},\ldots,\xi_{j_d}^{i_d})\prod_{l=1}^{d} a_{j_l}^{i_l},$$

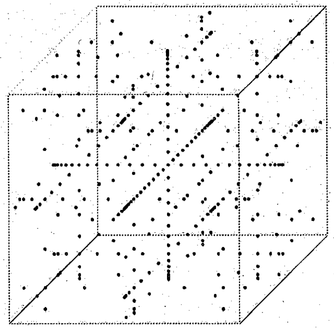
where $|\mathbf{i}| = \sum_{j=1}^{d} i_j$ and the coefficients a_j^i ($j = 1,\ldots,m_i$, $i = 1,\ldots,d$) are weights of one-dimensional quadrature rules.

Even if the one-dimensional weights are positive, some of these weights may become negative. Hence, an interpretation as discrete probability measure is no longer possible.

Convergence rates are very similar to those of QMC methods if the integrand f belongs to a tensor product Sobolev space.



(a) $d = 2$



(b) $d = 3$

Scenario reduction

We assume that the stochastic program behaves stable with respect to the Fortet-Mourier metric ζ_r.

Proposition: (Rachev-Rüschendorf 98)
If Ξ is bounded, ζ_r may be reformulated as transportation problem

$$\zeta_r(P,Q) = \inf\left\{\int_{\Xi\times\Xi} \hat{c}_r(\xi,\tilde{\xi})\eta(d\xi,d\tilde{\xi}) : \pi_1\eta = P, \pi_2\eta = Q\right\},$$

where $\hat{c}_r$ is a metric (reduced cost) with $\hat{c}_r \leq c_r$ and given by

$$\hat{c}_r(\xi,\tilde{\xi}) := \inf\left\{\sum_{i=1}^{n-1} c_r(\xi_{l_i},\xi_{l_{i+1}}) : n \in \mathbb{N}, \xi_{l_i} \in \Xi, \xi_{l_1} = \xi, \xi_{l_n} = \tilde{\xi}\right\}.$$

We consider discrete distributions P with scenarios ξ_i and probabilities p_i, $i = 1,\ldots,N$, and Q being supported by a given subset of scenarios ξ_j, $j \notin J \subset \{1,\ldots,N\}$, of P.

Best approximation given a scenario set J:

The best approximation of P with respect to ζ_r by such a distribution Q exists and is denoted by Q^*. It has the distance

$$D_J := \zeta_r(P, Q^*) = \min_Q \zeta_r(P, Q) = \sum_{i \in J} p_i \min_{j \notin J} \hat{c}_r(\xi_i, \xi_j)$$

and the probabilities $q_j^* = p_j + \sum_{i \in J_j} p_i$, $\forall j \notin J$, where

$J_j := \{i \in J : j = j(i)\}$ and $j(i) \in \arg\min_{j \notin J} \hat{c}_r(\xi_i, \xi_j)$, $\forall i \in J$

(optimal redistribution).

Determining the optimal index set J with prescribed cardinality $N - n$ is, however, a combinatorial optimization problem:

$$\min \{D_J : J \subset \{1, ..., N\}, |J| = N - n\}$$

Hence, the problem of finding the optimal set J for deleting scenarios is $\mathcal{NP}$-hard and polynomial time algorithms are not available.

$\longrightarrow$ Search for fast heuristics starting from $n = 1$ or $n = N - 1$.

Fast reduction heuristics

Starting point ($n = N-1$): $\min_{l \in \{1,\dots,N\}} p_l \min_{j \neq l} \hat{c}_r(\xi_l, \xi_j)$

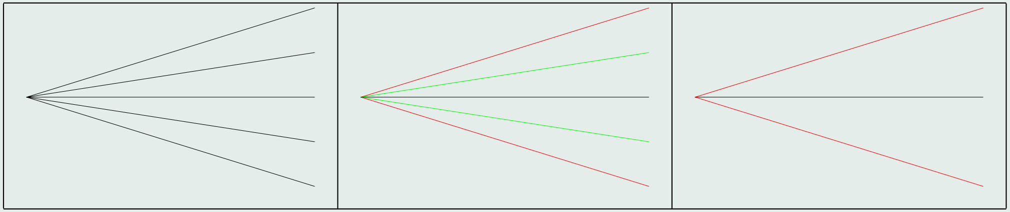
Algorithm 1: (Backward reduction)

Step [0]: $J^{[0]} := \emptyset\,.$

Step [i]: $l_i \in \arg \min_{l \notin J^{[i-1]}} \sum_{k \in J^{[i-1]} \cup \{l\}} p_k \min_{j \notin J^{[i-1]} \cup \{l\}} \hat{c}_r(\xi_k, \xi_j).$

$J^{[i]} := J^{[i-1]} \cup \{l_i\}\,.$

Step [N-n+1]: Optimal redistribution.



Starting point $(n=1)$: $\min_{u\in\{1,\dots,N\}} \sum_{k=1}^{N} p_k \hat{c}_r(\xi_k,\xi_u)$

Algorithm 2: (Forward selection)

Step [0]: $J^{[0]} := \{1,\dots,N\}$.

Step [i]: $$u_i \in \arg\min_{u\in J^{[i-1]}} \sum_{k\in J^{[i-1]}\setminus\{u\}} p_k \min_{j\notin J^{[i-1]}\setminus\{u\}} \hat{c}_r(\xi_k,\xi_j),$$

$$J^{[i]} := J^{[i-1]} \setminus \{u_i\}.$$

Step [n+1]: Optimal redistribution.

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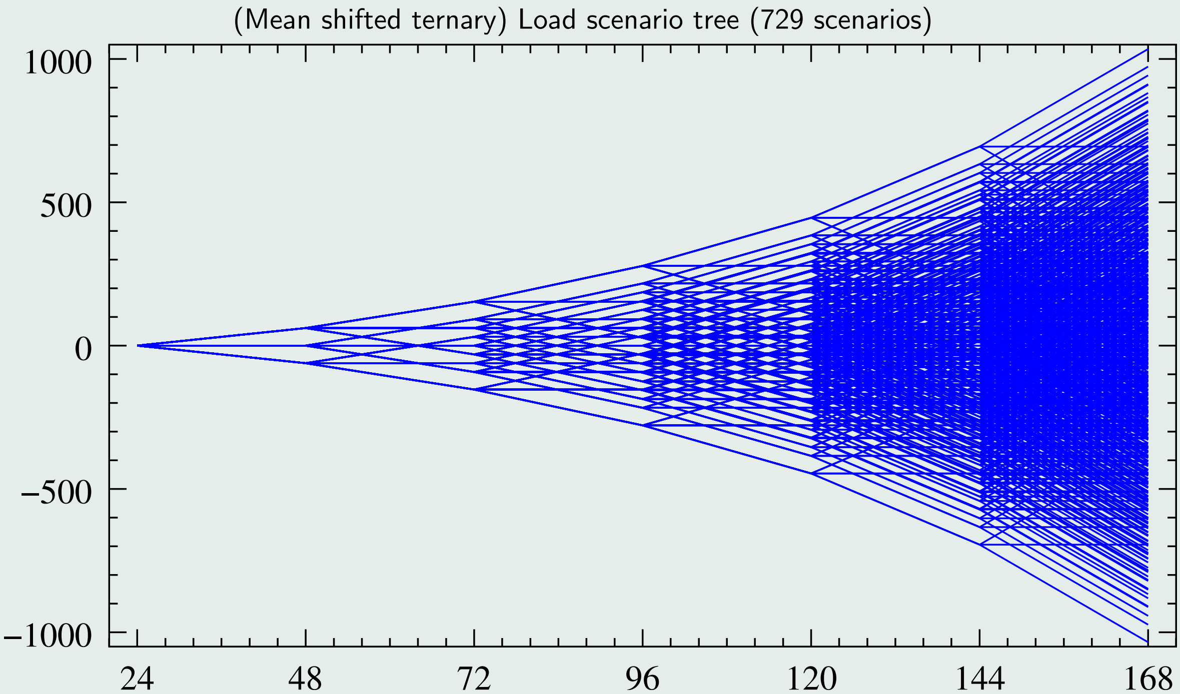
Example: (Electrical load scenario tree)



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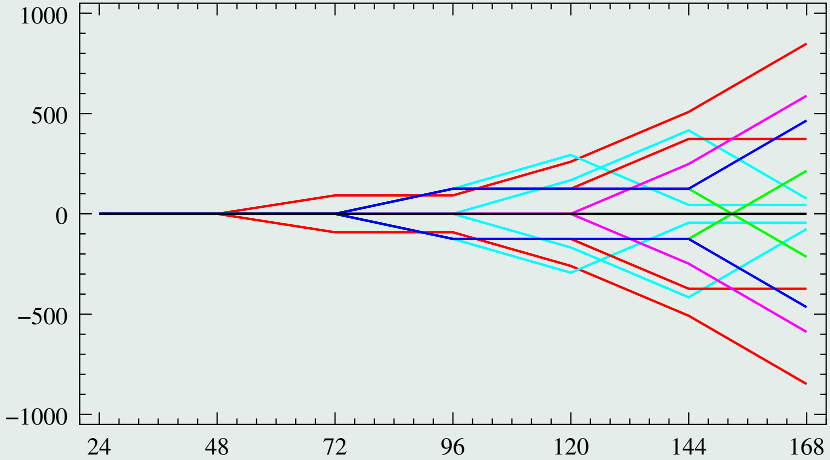
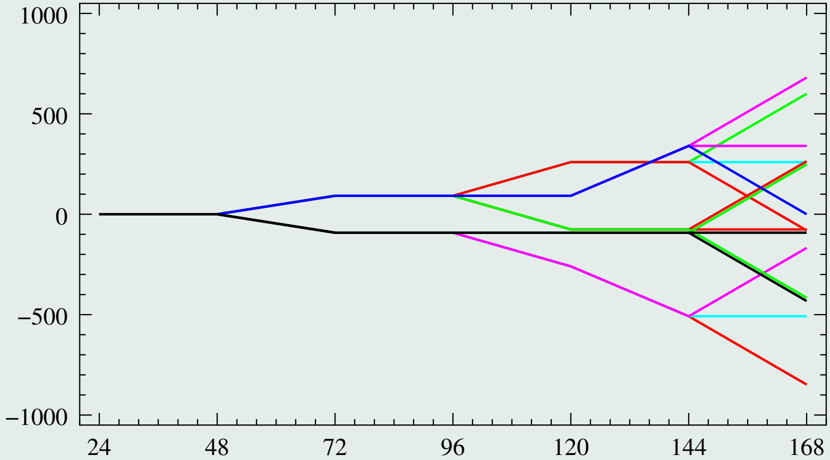
Reduced load scenario tree obtained by the forward selection method (15 scenarios)



Reduced load scenario tree obtained by the backward reduction method (12 scenarios)



Generation of scenario trees

Some recent approaches:

(1) Bound-based approximation methods: Kuhn 05, Casey-Sen 05.

(2) Monte Carlo-based schemes: Shapiro 03, 06.

(3) Quasi-Monte Carlo methods: Pennanen 06, 09 .

(4) Moment-matching principle: Høyland-Kaut-Wallace 03.

(5) Stability-based approximations: Hochreiter-Pflug 07, Mirkov-Pflug 07, Pflug-Pichler 10, Heitsch-Rö 05, 09.

Survey: Dupačová-Consigli-Wallace 00

Theoretical basis of (5): Stability results for multi-stage stochastic programs.

Scenario tree generation:

(i) Development of a stochastic model for the data process ξ (parametric [e.g. time series model], nonparametric [e.g. resampling from statistical data]) and generation of simulation scenarios;

(ii) Construction of a scenario tree out of the simulation scenarios by recursive scenario reduction and bundling over time such that the optimal expected revenue stays within a prescribed tolerance.

Implementation: GAMS-SCENRED 2.0 (by H. Heitsch)

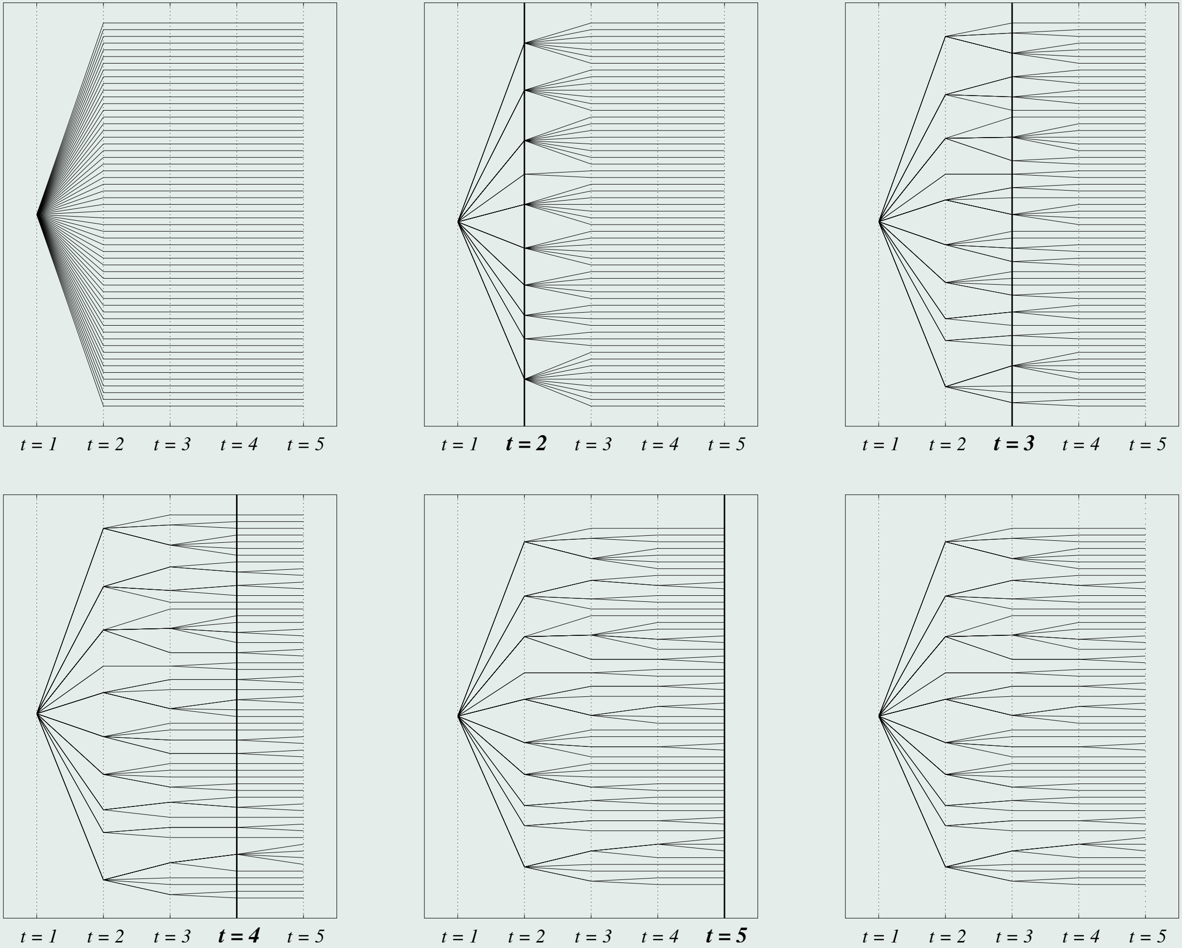


Illustration of the forward tree generation for an example including T=5 time periods starting with a scenario fan containing N=58 scenarios

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Appendix: Functions of bounded variation

Let $D = \{1, \ldots, d\}$ and we consider subsets u of D with cardinality $|u|$. By $-u$ we mean $-u = D \setminus u$.
The expression ξ^u denotes the $|u|$-tuple of the components ξ_j, $j \in u$, of $\xi \in \mathbb{R}^d$. For example, we write

$$f(\xi) = f(\xi^u, \xi^{-u}).$$

We set the d-fold alternating sum of f over the d-dimensional interval $[a, b]$ as

$$\triangle(f; a, b) = \sum_{u \subseteq D} (-1)^{|u|} f(a^u, b^{-u}).$$

Furthermore, we set for any $v \subseteq u$

$$\triangle_u(f; a, b) = \sum_{v \subseteq u} (-1)^{|v|} f(a^v, b^{-v}).$$

Let G_j denote finite grids in $[a_j, b_j)$, $a_j < b_j$, $j = 1, \ldots, d$, and $G = \times_{i=1}^d G_j$ a grid in $[a, b) = \times_{i=1}^d [a_j, b_j)$. For $g \in G$ let $g^+ = (g_1^+, \ldots, g_d^+)$, where g_j^+ is the successor of g_j in $G_j \cup \{b_j\}$. Then the variation of f over G is

$$V_G(f) = \sum_{g \in G} \left| \triangle(f; g, g^+) \right| .$$

If $\mathcal{G}$ denotes the set of all finite grids in $[a, b)$, the variation of f on $[a, b]$ in the sense of Vitali is

$$V_{[a,b]}(f) = \sup_{G \in \mathcal{G}} V_G(f) .$$

The variation of f on $[a, b]$ in the sense of Hardy and Krause is

$$V_{\mathrm{HK}}(f; a, b) = \sum_{u \subset D} V_{[a^{-u}, b^{-u}]}(f(\xi^{-u}, b^u)) .$$

Bounded variation on $[a, b]$ in the sense of Hardy and Krause then means $V_{\mathrm{HK}}(f; a, b) < \infty$.

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